

# ON A TIME AND SPACE DISCRETIZED APPROXIMATION OF THE BOLTZMANN EQUATION IN THE WHOLE SPACE

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**ABSTRACT.** In this paper, convergence results on the solutions of a time and space discrete model approximation of the Boltzmann equation for a gas of Maxwellian particles in a bounded domain, obtained by Babovsky and Illner [1989], are extended to approximate the solutions of the Boltzmann equation in the whole physical space. This is done for a class of particle interactions including Maxwell and soft cut-off potentials in the sense of Grad.

The main result shows that the solutions of the discrete model converge in  $L^1$  to the solutions of the Boltzmann equation, when the discretization parameters go simultaneously to zero. The convergence is uniform with respect to the discretization parameters.

In addition, a sufficient condition for the implementation of the main result is provided.

## 1. INTRODUCTION

In a known paper [1], Babovsky and Illner provided a validation (convergence) proof of Nanbu's simulation method [2] for the spatially inhomogeneous (full) Boltzmann equation [4, 5] describing a rarefied gas of Maxwellian particles confined to a bounded spatial domain (with specularly reflecting boundary conditions). More specifically, the main result (Theorem 7.1) of [1] demonstrated that the discrete measures provided by Nanbu's simulation method are almost surely weakly convergent to absolutely continuous measures with densities given by solutions of the Boltzmann equation.

In essence, the analysis behind the main theorem of [1] represented a space-dependent generalization of the convergence proof of Nanbu's simulation algorithm for the space-homogeneous Boltzmann equation, provided in an earlier work by Babovsky [6]. Briefly, in [1], the space-homogeneous simulation algorithm of [6] was applied to a suitable time and space discrete Boltzmann model (Eq. (5.14) in [1]). The latter was derived from the Boltzmann equation by means of time discretization, splitting (separation of free flow and collisional interactions), cell-partitioning of the physical domain of the gas, and space-averaging (homogenization) over cells. The discretization was parameterized by a time-step and an upper bound for the maximum of all cell diameters. The analysis was completed by combining convergence properties of the discrete Boltzmann model with those of the space-homogeneous simulation algorithm of [6]. To this end, Babovsky and

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2010 *Mathematics Subject Classification.* 35A35, 65M12, 76P05.

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Illner established the key result (Corollary 5.1 in [1]) that the solutions of the discrete Boltzmann model converge in discrepancy<sup>1</sup> to the solutions of the Boltzmann equation for the gas in a bounded spatial volume, uniformly with respect to the parameters of the discretization, when these parameters converge simultaneously to zero.

A notable thing about the proof of the convergence in discrepancy of the solutions of the discretized Boltzmann model is that, as it appears in [1], is responsible for the limitation of the analysis of [1] to the case of the Boltzmann gas in a bounded spatial domain. Indeed, the boundedness of the spatial domain was actually assumed in [1] in order to prove the above key convergence result for the discrete Boltzmann equation (see [1], p. 59). An alternative proof, without the boundedness assumption, might allow the conclusions of [1] to be extended to other important examples, e.g., a gas expanding in the whole physical space.

In this paper, the results on the convergence in discrepancy established by Babovsky and Illner for the solutions of their discrete Boltzmann model of [1] are extended to the setting of the Boltzmann equation in the entire physical space. More specifically, in such a setting, we show that the solutions of the discrete Boltzmann model converge in  $\mathbb{L}^1$  to the solutions of the Boltzmann equation in the whole physical space, uniformly with respect to the parameters of the discretization, when these parameters converge simultaneously to zero. We also show that the solutions of the discrete approximation satisfy the conservation laws for mass, momentum and energy.

Here, it should be recalled that the results of [1] concern the Boltzmann equation for Maxwellian particles. The limitation to Maxwellian interactions does not come from the proof of the analytical convergence of the discretized Boltzmann model, but is imposed by the implementation of the simulation algorithm of [6] for the validation of Nanbu's scheme (see [1], p. 48). However, besides its usefulness in the validation of the Nanbu's scheme, the discrete Boltzmann model of [1] might be applied to obtain new (not necessarily probabilistic) rigorous algorithms for the Boltzmann equation. Thus, understanding its convergence properties in more general situations than in [1] may be of interest. In this respect, as an additional contribution, our main result concerns the Boltzmann equation with Maxwell and soft cut-off collision kernels in the sense of Grad [3].

Compared to [1], our analysis must face additional difficulties, since one has to estimate, uniformly, in some sense, how high speed gas particles situated at large distances contribute to the gas evolution, approximated as in [1], by an alternation of molecular transport and collision steps. In this respect, a technical point is reconsidering the important property established by Babovsky and Illner (Theorem 5.1 in [1]) that, under suitable conditions, if the Boltzmann equation is approximated by the discrete Boltzmann model, then the family of errors introduced by the approximation is bounded in some  $\mathbb{L}^\infty$  - (velocity) Maxwellian weighted space, uniformly with respect to the parameters of the discretization. This property was demonstrated in [1], in the setting of the Boltzmann equation in a bounded spatial domain, but remains actually valid in a larger context, as is implicit from [1]. Nevertheless, for the sake of clarity and completeness, in the present work, we

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<sup>1</sup>Let  $\mu$  and  $\nu$  be two (Borel) probability measures on the same measure space  $\mathfrak{B} \subseteq \mathbb{R}^n$ . Consider  $\mathfrak{B}$  with the usual semi-order  $\leq$  of  $\mathbb{R}^n$ . Following [1], the discrepancy between  $\mu$  and  $\nu$  is defined as  $D(\mu, \nu) := \sup_{z \in \mathbb{R}^n} |\mu\{y \in \mathfrak{B} : y \leq z\} - \nu\{y \in \mathfrak{B} : y \leq z\}|$ .

will prove a precise statement appropriate to our framework (see Proposition 1 in Subsection 4.2).

The rest of this paper is structured as follows. In Section 2, we present the discrete Boltzmann model of [1], and formally introduce the main result. However, a precise formulation (Theorem 1) is given in the second part of Section 3. This requires some preparation in the first part of the same section. The second part of Section 3 also includes Theorem 2 which provides sufficient conditions for the application of Theorem 1. Section 4 deals with the proofs of the theorems stated in Section 3. The proofs rely on technical estimates provided in Subsection 4.1. In particular, standard  $\mathbb{L}^\infty$  - type inequalities for the collision term are adapted to our setting, supplemented with useful  $\mathbb{L}^1$  - estimates. The central result of Subsection 4.1 is Lemma 4, needed later to measure, in some sense, the errors introduced when the discrete Boltzmann model approximates the Boltzmann equation. The results of Subsection 4.1 are then used in Subsection 4.2 to prove Proposition 1, ultimately leading to the proof of Theorem 1. Subsection 4.3 contains the proof of Theorem 2. Finally, Section 5 presents a simple application to a Boltzmann model for a rarefied gas expanding in the whole space, and closes with a few concluding remarks and possible future directions.

## 2. DISCRETIZED BOLTZMANN MODEL FOR THE BOLTZMANN EQUATION

In this section we recall some very basic facts about the Boltzmann equation, and briefly present its time and space discrete approximation of [1], adapted to our setting. Finally, we formally introduce our results.

The Cauchy problem for the Boltzmann equation for a simple gas (monatomic gas of identical particles with elastic binary collisions), evolving in the whole physical space reads (in non-dimensional units) as [4, 5]

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = J(f) \quad \text{in } (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (1)$$

$$f(0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}) \quad \text{on } \mathbb{R}^3 \times \mathbb{R}^3,$$

where the unknown  $f = f(t, \mathbf{x}, \mathbf{v}) \geq 0$  represents the distribution density of the gas particles at time  $0 \leq t < T \leq \infty$ , with position  $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ , and velocity  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ . The right hand side of the above equation is the nonlinear Boltzmann collision term  $J(f) := J_B(f, f)$ , where

$$J_B(g, h)(t, \mathbf{x}, \mathbf{v}) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\omega})(g' h'_* - g h_*) d\mathbf{v}_* d\boldsymbol{\omega}. \quad (2)$$

acts on  $g$  and  $h$  only through the velocity dependence. Here, we are using the standard shorthand notations  $g := g(t, \mathbf{x}, \mathbf{v})$ ,  $h_* := h(t, \mathbf{x}, \mathbf{v}_*)$ ,  $g' := g(t, \mathbf{x}, \mathbf{v}')$ ,  $h'_* := h(t, \mathbf{x}, \mathbf{v}'_*)$ . Moreover,

$$\mathbf{v}' = \mathbf{v} - ((\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\omega}) \boldsymbol{\omega}, \quad \mathbf{v}'_* = \mathbf{v}_* + ((\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\omega}) \boldsymbol{\omega}, \quad (3)$$

are the post-collisional velocities expressed, in terms of the pre-collisional velocities  $(\mathbf{v}, \mathbf{v}_*)$  and the collision parameter  $\boldsymbol{\omega}$  over the unit sphere  $\mathbb{S}^2$ , as (parameterized) solutions of the laws of momentum and energy conservation in binary elastic collisions

$$\mathbf{v} + \mathbf{v}_* = \mathbf{v}' + \mathbf{v}'_*, \quad \mathbf{v}^2 + \mathbf{v}_*^2 = \mathbf{v}'^2 + \mathbf{v}'_*^2. \quad (4)$$

Furthermore, in (2), the collision kernel  $b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\omega})$  is a given nonnegative function that depends only on the modulus of the relative velocity  $|\mathbf{v} - \mathbf{v}_*|$  and the scalar product  $(\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\omega}$ .

A useful decomposition of (2) is  $J_B(g, h) = P_B(g, h) - S_B(g, h)$  where

$$\begin{aligned} P_B(g, h)(\mathbf{x}, \mathbf{v}) &:= \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\omega}) g' h'_* d\mathbf{v}_* d\boldsymbol{\omega}, \\ S_B(g, h)(\mathbf{x}, \mathbf{v}) &:= \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\omega}) g h_* d\mathbf{v}_* d\boldsymbol{\omega}, \end{aligned} \quad (5)$$

provided that the integrals are well defined. We refer to [5] for a comprehensive presentation of the Boltzmann equation and its applications. Here we just recall the property

$$\int_{\mathbb{R}^3} \varphi_i(\mathbf{v}) J(g)(\mathbf{x}, \mathbf{v}) d\mathbf{v} = 0, \quad i = 0, 1, 2, 3, 4, \quad (6)$$

valid for any  $g$  for which the above integral exists, where  $\varphi_0(\mathbf{v}) := 1$ ,  $\varphi_i(\mathbf{v}) := v_i$ ,  $i = 1, 2, 3$ ,  $\varphi_4(\mathbf{v}) := |\mathbf{v}|^2$ . Formally, this implies that the solution of the Boltzmann equation satisfies the fluid balance laws for mass, momentum and energy [5].

Let  $0 < \Delta t < T$  be a discretization time-step for the time interval  $[0, T]$ . Suppose that  $\mathbb{R}^3$  is partitioned into a countable family  $\Pi := (\pi_l)_{l \in \mathbb{N}}$  of distinct cells  $\pi_l$  with finite diameter  $D(\pi_l)$ . Then the discrete model provided in [1] to approximate the Boltzmann equation in a bounded domain (Eq. (5.14) in [1]) can be reformulated in our setting (to approximate (1)) as

$$\begin{cases} \tilde{f}^0(\mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}), \\ \tilde{f}^j(\mathbf{x}, \mathbf{v}) = U^{\Delta t} \tilde{f}^{j-1}(\mathbf{x}, \mathbf{v}) + \Delta t J^\pi(U^{\Delta t} \tilde{f}^{j-1})(\mathbf{x}, \mathbf{v}), \quad j = 1, \dots, \lfloor \frac{T}{\Delta t} \rfloor. \end{cases} \quad (7)$$

Here,  $U^t$  is the free streaming operator

$$(U^t g)(\mathbf{x}, \mathbf{v}) := g(\mathbf{x} - t\mathbf{v}, \mathbf{v}), \quad t \in \mathbb{R}, \quad (8)$$

$J^\pi$  is the homogenized Boltzmann operator

$$J^\pi(g) := J_B(g, \pi g), \quad (9)$$

defined by means of the operator of homogenization of the space cells

$$(\pi g)(\mathbf{x}, \mathbf{v}) := \sum_{l \in \mathbb{N}} \chi_{\pi_l}(\mathbf{x}) \frac{1}{|\pi_l|} \int_{\pi_l} g(\mathbf{y}, \mathbf{v}) d\mathbf{y}, \quad (10)$$

where  $\chi_{\pi_l}$  is the indicator function of the cell  $\pi_l$ ,  $|\pi_l|$  denotes the volume of the cell. Moreover,  $\lfloor \cdot \rfloor$  is the integer part function.

Apart from minor changes of notation, the main difference between (7) and the formulation of the discretized Boltzmann model of [1] is that the sum in (7) is infinite, because, in our setting, we are dealing with the partitioning of the whole  $\mathbb{R}^3$  into finite cells.

In what follows, we shall always assume that  $\Pi := (\pi_l)_{l \in \mathbb{N}}$  has a finite (partition) diameter

$$\Delta \mathbf{x} = \Delta \mathbf{x}(\Pi) := \sup_{l \in \mathbb{N}} D(\pi_l) < \infty. \quad (11)$$

We are interested in the convergence properties of the solutions of (7).

More specifically, we investigate the convergence properties of

$$f_{\Delta t, \Delta \mathbf{x}}(t, \mathbf{x}, \mathbf{v}) := \sum_{j=1}^{\lfloor \frac{T}{\Delta t} \rfloor} \chi_j(t) \tilde{f}^j(\mathbf{x}, \mathbf{v}), \quad 0 \leq t < T, \quad (12)$$

where  $\chi_j$  is the indicator function of the real interval  $[(j-1)\Delta t, j\Delta t)$ .

In this paper, we suppose that (1) concerns a gas with finite total mass, hence physically meaningful solutions of the equation are elements of the positive cone  $\mathbb{L}_+^1$  of  $\mathbb{L}^1 := \mathbb{L}^1(\mathbb{R}^3 \times \mathbb{R}^3; d\mathbf{x}d\mathbf{v})$  – real. Therefore, our results will be formulated in the sense of the convergence in  $\mathbb{L}^1$ .

Our basic hypotheses is that the collision kernel satisfies Grad's soft cutoff condition [3]. More precisely, throughout, we maintain the following assumption:

**Assumption 1.** *There exist two constants  $b_0 > 0$  and  $0 \leq \lambda < 2$  such that*

$$\int_{\mathbb{S}^2} b(|\mathbf{v} - \mathbf{v}_*|, \omega) d\omega \leq b_0 |\mathbf{v} - \mathbf{v}_*|^{-\lambda}. \quad (13)$$

Remark that the collision kernel for Maxwellian molecules corresponds to the particular case  $\lambda = 0$  in (13).

The main result of this paper shows that if  $f(t) \in \mathbb{L}_+^1$  is a solution to the Cauchy problem (1), which decays in positions and velocities at infinity, and has suitable Lipschitz regularity properties (similar to those required in [1]), then  $f_{\Delta t, \Delta \mathbf{x}}(t) := f_{\Delta t, \Delta \mathbf{x}}(t, \cdot, \cdot) \in \mathbb{L}_+^1$  and  $f_{\Delta t, \Delta \mathbf{x}}(t) \xrightarrow{\mathbb{L}^1} f(t)$  as  $(\Delta t + \Delta \mathbf{x}) \rightarrow 0$ , uniformly in  $\Delta t$  and  $\Delta \mathbf{x}$ . In addition, the approximation  $f_{\Delta t, \Delta \mathbf{x}}$  is consistent with the laws of global conservation for mass, momentum and energy, respectively,

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{x}d\mathbf{v} \varphi_i(\mathbf{v}) f_{\Delta t, \Delta \mathbf{x}}(t, \mathbf{x}, \mathbf{v}) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{x}d\mathbf{v} \varphi_i(\mathbf{v}) f_0(\mathbf{x}, \mathbf{v}), \quad (14)$$

where  $\varphi_i(\mathbf{v})$ ,  $i = 0, 1, 2, 3, 4$  are as in (6).

### 3. MAIN RESULT

**3.1. Basic notations and definitions.** In general, in a given context of this paper, different constants are differently denoted. If some constant  $c$  depends on parameters  $\alpha_1, \alpha_2, \dots$ , we will also denote it by  $c_{\alpha_1, \alpha_2, \dots}$ , in order to make explicit its parameter dependence.

Let  $0 \leq \alpha < \infty$ ,  $0 < \tau < \infty$  and  $m_{\alpha, \tau}(\mathbf{x}, \mathbf{v}) := \exp(-\alpha \mathbf{x}^2 - \tau \mathbf{v}^2)$ .

We consider the following  $\mathbb{L}^\infty$  - weighted spaces. By  $\mathbf{B}_\tau$ , we denote the subspace of elements  $g \in \mathbb{L}^\infty := \mathbb{L}^\infty(\mathbb{R}^3 \times \mathbb{R}^3; d\mathbf{x}d\mathbf{v})$  – real satisfying  $\|g\|_{\mathbf{B}_\tau} := \|m_{0, \tau}^{-1} g\|_{\mathbb{L}^\infty} < \infty$ . Moreover,  $\mathbf{M}_\tau$  denotes the subspace of the elements  $g \in \mathbf{B}_\tau$  with  $\|g\|_{\mathbf{M}_\tau} := \|m_{\tau, \tau}^{-1} g\|_{\mathbb{L}^\infty} < \infty$ . Generically, we will refer to  $\|\cdot\|_{\mathbf{B}_\tau}$  and  $\|\cdot\|_{\mathbf{M}_\tau}$  as the  $\mathbf{B}$  - norm and  $\mathbf{M}$  - norm, respectively. We denote by  $\mathbf{B}_{\tau, +}$  and  $\mathbf{M}_{\tau, +}$  the positive cones of  $\mathbf{B}_\tau$  and  $\mathbf{M}_\tau$ , respectively, considered with the natural order  $\leq$  (induced by  $\mathbb{L}^\infty$ ).

Notice that (8) defines groups of linear isometries  $\{U^t\}_{t \in \mathbb{R}}$  on  $\mathbf{B}_\tau$  and  $\mathbb{L}^1$ ,

$$\begin{aligned} \|U^t g\|_{\mathbf{B}_\tau} &= \|g\|_{\mathbf{B}_\tau}, \quad \forall g \in \mathbf{B}_\tau, \quad t \in \mathbb{R}, \\ \|U^t g\|_{\mathbb{L}^1} &= \|g\|_{\mathbb{L}^1}, \quad \forall g \in \mathbb{L}^1, \quad t \in \mathbb{R}. \end{aligned} \quad (15)$$

One can easily check that if  $0 < \tau_1, T < \infty$ , then there is some  $\Theta = \Theta(T, \tau_1) < \tau_1$ , depending on  $T$  and  $\tau_1$ , such that for any  $g \in \mathbf{M}_{\tau_1}$ , we have

$$\|U^t g\|_{\mathbf{M}_\tau} \leq \|g\|_{\mathbf{M}_{\tau_1}}, \quad 0 \leq |t| \leq T, \quad 0 \leq \tau < \Theta. \quad (16)$$

One can also observe that (10) defines  $\pi$  as a linear contraction both in  $\mathbf{B}_\tau$  and  $\mathbb{L}^1$ :

$$\begin{aligned} \|\pi g\|_{\mathbf{B}_\tau} &\leq \|g\|_{\mathbf{B}_\tau}, \quad \forall g \in \mathbf{B}_\tau, \\ \|\pi g\|_{\mathbb{L}^1} &\leq \|g\|_{\mathbb{L}^1}, \quad \forall g \in \mathbb{L}^1. \end{aligned} \quad (17)$$

Further, on  $\mathbf{B}_\tau$ , we consider the group of spatial translations  $\{T_{\mathbf{y}}\}_{\mathbf{y} \in \mathbb{R}^3}$  defined by  $(T_{\mathbf{y}}g)(\mathbf{x}, \mathbf{v}) := g(\mathbf{x} + \mathbf{y}, \mathbf{v})$  for almost all  $\mathbf{x}, \mathbf{v}$ . Obviously,

$$\|T_{\mathbf{y}}g\|_{\mathbf{B}_\tau} = \|g\|_{\mathbf{B}_\tau}, \quad \mathbf{y} \in \mathbb{R}^3, \quad (18)$$

and

$$(U^t g)(\mathbf{x}, \mathbf{v}) = (T_{-(t\mathbf{v})}g)(\mathbf{x}, \mathbf{v}), \quad t \in \mathbb{R}, \quad (19)$$

for almost all  $\mathbf{x}, \mathbf{v} \in \mathbb{R}^3$ .

Clearly,  $U^t$ ,  $\pi$  and  $T_{\mathbf{y}}$  preserve the order, in particular the positivity, in  $\mathbf{B}_\tau$ ,  $\mathbf{M}_\tau$  and  $\mathbb{L}^1$ .

We need to introduce some suitable subsets of  $\mathbf{M}_\tau$  on which the map  $\mathbf{y} \rightarrow T_{\mathbf{y}}$  is Lipschitz continuous.

Let  $0 < R, M < \infty$ . By  $\mathcal{M}_\tau(R, M)$ , we denote the subset of the elements  $g \in \mathbf{M}_\tau$  satisfying

$$\|g\|_{\mathbf{M}_\tau} \leq R, \quad (20)$$

and

$$\|(T_{\mathbf{y}}g) - g\|_{\mathbf{M}_\tau} \leq M|\mathbf{y}|, \quad |\mathbf{y}| \leq 1. \quad (21)$$

It follows that  $\mathcal{M}_\tau(R, M) \subset \mathbf{M}_\tau \subset \mathbb{L}^1 \cap \mathbf{B}_\tau$ . Moreover, straightforward calculations imply that, for any  $g \in \mathcal{M}_\tau(R, M)$ , we have

$$\|g\|_{\mathbf{B}_\tau} \leq \|g\|_{\mathbf{M}_\tau} \leq R, \quad (22)$$

$$\|g\|_{\mathbb{L}^1} \leq \left(\frac{\pi}{\tau}\right)^3 \|g\|_{\mathbf{M}_\tau} \leq \left(\frac{\pi}{\tau}\right)^3 R, \quad (23)$$

$$\|\pi g - g\|_{\mathbf{B}_\tau} \leq M\Delta\mathbf{x}, \quad \Delta\mathbf{x} \leq 1, \quad (24)$$

and

$$\|\pi g - g\|_{\mathbb{L}^1} \leq \left(\frac{\pi}{\tau}\right)^3 M\Delta\mathbf{x}, \quad \Delta\mathbf{x} \leq 1. \quad (25)$$

We set  $\mathcal{M}_{\tau,+}(R, M) := \{g \in \mathcal{M}_\tau(R, M) : g \geq 0\}$ .

**3.2. Main theorem and a practical criterion.** We are interested in approximating the  $\mathbb{L}^1$  - mild solutions of (1). These are solutions of the equation (see e.g., [10])

$$f(t) = U^t f_0 + \int_0^t U^{t-s} J(f(s)) ds, \quad (26)$$

the integration with respect to  $ds$  being in the sense of Riemann in  $\mathbb{L}^1$ . Here, the nonlinear operator  $J$  is defined on its natural domain of  $\mathbb{L}^1$  by (2) (with  $b(|\mathbf{v} - \mathbf{v}'|, \boldsymbol{\omega})$  satisfying assumption (13)).

**Definition 1.** Let  $T > 0$ . An element  $f \in C(0, T; \mathbb{L}^1)$  is called mild solution on  $[0, T]$  for the Cauchy problem (1) in  $\mathbb{L}^1$ , if it satisfies Eq. (26).

Now we are in position to formulate our main result.

Let  $0 < \tau, T, R, M < \infty$ , and denote by  $\Lambda$  the couple of parameters  $(b_0, \lambda)$  appearing in Assumption 1.

Consider  $f_{\Delta t, \Delta \mathbf{x}}(t)$  as in (12).

**Theorem 1.** *Suppose that the Cauchy problem (1) has a mild solution  $f(t) \in \mathcal{M}_{\tau,+}(R, M)$ ,  $0 \leq t \leq T$ . Then, for each  $0 < \sigma < \tau$ , there are some numbers  $0 < X_* < 1$ ,  $0 < T_* < \min(1, T)$ , and  $0 < K_i < \infty$ ,  $i = 1, 2$ , depending on  $R, M, T, \tau, \sigma$  and  $\Lambda$ , such that, for any  $0 < \Delta t \leq T_*$  and  $0 < \Delta \mathbf{x} \leq X_*$ , one has*

$$f_{\Delta t, \Delta \mathbf{x}}(t) \in \mathbb{L}_+^1 \cap \mathbf{B}_\sigma, \quad 0 \leq t < T \quad (27)$$

and

$$\sup_{t \in [0, T]} \|f_{\Delta t, \Delta \mathbf{x}}(t) - f(t)\|_{\mathbb{L}^1} \leq K_1 \Delta t + K_2 \Delta \mathbf{x}. \quad (28)$$

Moreover,  $f_{\Delta t, \Delta \mathbf{x}}(t)$  satisfies property (14) for all  $0 \leq t < T$ .

The proof of the theorem will be given in Subsection 4.2. We only remark here that the proof applies estimates based on the decomposition  $J := P - S$  into the standard gain and loss operators [5, 10], respectively, as well as estimates based on the similar decomposition  $J^\pi = P^\pi - S^\pi$ , where

$$P(g) := P_B(g, g), \quad S(g) := S_B(g, g)$$

and

$$P^\pi(g) := P_B(g, \pi g), \quad S^\pi(g) := S_B(g, \pi g),$$

with  $P_B$  and  $S_B$  given by (5). Due to assumption (13), the above expressions define  $P, S, P^\pi$  and  $S^\pi$  as locally Lipschitz continuous positive maps in  $\mathbf{B}_\tau$  (in  $\mathbf{M}_\tau$ ).

A few remarks are in order.

Since  $P, S$  are locally Lipschitz continuous operators in  $\mathbf{B}_\tau$ , and  $\{U^t\}_{t \in \mathbb{R}}$  is a continuous group of isometries on  $\mathbf{B}_\tau$ , the contraction mapping principle applied to Eq. (26) implies easily the existence and uniqueness of local-in-time solutions (at least) in  $\mathbf{B}_\tau$  (see, e.g., [3] for the case of Maxwellian molecules). In fact, there are more general results on the local and global existence, uniqueness, and positivity of solutions to Eq. (26), in various spaces of functions decaying in velocities and positions at infinity [7, 8, 9, 10, 11, 12, 13].

We end this section with a sufficient condition for the applicability of Theorem 1, which allows for replacing the Lipschitz assumption imposed in Theorem 1 to the solutions of Eq. (26) by a similar one on the initial data.

**Theorem 2.** *Let  $0 < \tau_*, R, T < \infty$ . Suppose that the Cauchy problem (1) has a mild solution  $f(t) \in \mathbf{M}_{\tau_*,+}$  such that*

$$\|f(t)\|_{\mathbf{M}_{\tau_*}} \leq R, \quad 0 \leq t \leq T. \quad (29)$$

*If there is a constant  $M_0 > 0$  such that*

$$f_0 \in \mathcal{M}_{\tau_*}(R, M_0), \quad (30)$$

*then, for any  $0 < \tau_1 < \tau_*$ , there are two numbers,  $0 < \tau < \tau_*$  (depending on  $T, \tau_*$  and  $\tau_1$ ) and  $0 < M < \infty$  (depending on  $R, M_0, T, \tau_*, \tau_1$  and  $\Lambda$ ), such that*

$$f(t) \in \mathcal{M}_\tau(R, M), \quad 0 \leq t \leq T. \quad (31)$$

The proof of the theorem is given in Subsection 4.3.

It should be observed that the subset of elements of  $\mathbf{M}_{\tau_*}$  satisfying (30) for some  $M_0 > 0$  is rather large, in the sense that  $\cup_{M_0 > 0} \mathcal{M}_{\tau_*}(R, M_0)$  it is dense in  $\{g \in \mathbb{L}^1 : \|g\|_{\mathbf{M}_{\tau_*}} \leq R\}$  (with respect to the topology of  $\mathbb{L}^1$ ).

#### 4. TECHNICAL PROOFS

**4.1. Auxiliary estimates.** In the following, some useful  $\mathbf{B}$  - norm inequalities of [1], related to the properties of (7) and (26) are extended to our setting. These are then supplemented with new  $\mathbb{L}^1$  - norm analogous inequalities. The main purpose is to prove Lemma 4 which will play a key role in the next subsection, in estimating the cumulative effect of the “errors” introduced when the solutions of the Boltzmann equation are approximated by the solutions of (7).

The next lemma and its immediate consequences are based on standard estimations on the collision operator, similar to those in [1, 8, 13, 14]. To formulate the lemma, first denote

$$G(\mathbf{v}; \tau, \lambda) := \int_{\mathbb{R}^3} |\mathbf{v} - \mathbf{v}_*|^{-\lambda} \exp(-\tau \mathbf{v}_*^2) d\mathbf{v}_*; \quad 0 \leq \lambda < 2, \quad 0 < \tau < \infty, \quad (32)$$

and recall that by integrating upon  $\mathbf{v}_*$ , in cylindrical coordinates in a frame with the  $v_{*,3}$  axis along the direction of  $\mathbf{v}$ , one gets (see, e.g., [8])

$$\sup_{\mathbf{v} \in \mathbb{R}^3} G(\mathbf{v}; \tau, \lambda) \leq \pi^{\frac{3}{2}} \Pi\left(\frac{-\lambda}{2}\right) \tau^{-\frac{3-\lambda}{2}}, \quad (33)$$

where  $\Pi(z)$  is the Gauss’ Pi function.

With  $b_0 > 0$  as in (13), set

$$b_\Lambda := \Pi\left(\frac{-\lambda}{2}\right) b_0, \quad (34)$$

$$c_{\Lambda, \tau} = 2\pi^{\frac{3}{2}} \Pi\left(\frac{-\lambda}{2}\right) \tau^{-\frac{3-\lambda}{2}} b_0,$$

as constants to be frequently used in the sequel.

**Lemma 1.** *a) For any  $m_{\alpha, \tau}^{-1} g_i, m_{\alpha, \tau}^{-1} h_i \in \mathbb{L}^\infty$ ,  $i = 1, 2$ , one has*

$$\begin{aligned} & \|m_{\alpha, \tau}^{-1} [J_B(g_1, g_2) - J_B(h_1, h_2)]\|_{\mathbb{L}^\infty} \\ & \leq c_{\Lambda, \tau} (\|m_{\alpha, \tau}^{-1} g_1\|_{\mathbb{L}^\infty} \|m_{0, \tau}^{-1} (g_2 - h_2)\|_{\mathbb{L}^\infty} + \|m_{\alpha, \tau}^{-1} h_2\|_{\mathbb{L}^\infty} \|m_{0, \tau}^{-1} (g_1 - h_1)\|_{\mathbb{L}^\infty}). \end{aligned} \quad (35)$$

*b) For any  $g_i, h_i \in \mathbb{L}^1 \cap \mathbf{B}_\tau$ ,  $i = 1, 2$ , one has*

$$\|J_B(g_1, g_2) - J_B(h_1, h_2)\|_{\mathbb{L}^1} \leq c_{\Lambda, \tau} (\|g_1\|_{\mathbf{B}_\tau} \|g_2 - h_2\|_{\mathbb{L}^1} + \|h_2\|_{\mathbf{B}_\tau} \|g_1 - h_1\|_{\mathbb{L}^1}). \quad (36)$$

**Proof:** Since  $J_B = P_B - S_B$ , the proof follows from suitable norm estimations of  $S_B$  and  $P_B$ , respectively. To this end, first observe that by applying (13) in (5), one can write

$$\begin{aligned} & |S_B(g_1, g_2)(\mathbf{x}, \mathbf{v}) - S_B(h_1, h_2)(\mathbf{x}, \mathbf{v})| \\ & \leq b_0 \int_{\mathbb{R}^3} |\mathbf{v} - \mathbf{v}_*|^{-\lambda} (|g_1| |g_2 - h_2| + |h_2| |g_1 - h_1|) d\mathbf{v}_* \end{aligned} \quad (37)$$



and

$$\begin{aligned} & |P_B(g_1, g_2)(\mathbf{x}, \mathbf{v}) - P_B(h_1, h_2)(\mathbf{x}, \mathbf{v})| \\ & \leq b_0 \int_{\mathbb{R}^3} |\mathbf{v} - \mathbf{v}_*|^{-\lambda} (|g'_1| |g'_{2*} - h'_{2*}| + |h'_{2*}| |g'_1 - h'_1|) d\mathbf{v}_*. \end{aligned} \quad (38)$$

a) For the moment, let us denote the right hand side of (35) by

$$Q_{g_1, g_2}^{h_1, h_2} = c_{\Lambda, \tau} (\|m_{\alpha, \tau}^{-1} g_1\|_{\mathbb{L}^\infty} \|m_{0, \tau}^{-1} (g_2 - h_2)\|_{\mathbb{L}^\infty} + \|m_{\alpha, \tau}^{-1} h_2\|_{\mathbb{L}^\infty} \|m_{0, \tau}^{-1} (g_1 - h_1)\|_{\mathbb{L}^\infty}).$$

In the integrand of (37), we introduce the obvious inequalities

$$\begin{aligned} |g_1(\mathbf{x}, \mathbf{v})| & \leq m_{\alpha, \tau}(\mathbf{x}, \mathbf{v}) \|m_{\alpha, \tau}^{-1} g_1\|_{\mathbb{L}^\infty}, \quad |h_2(\mathbf{x}, \mathbf{v}_*)| \leq m_{\alpha, \tau}(\mathbf{x}, \mathbf{v}_*) \|m_{\alpha, \tau}^{-1} h_2\|_{\mathbb{L}^\infty}, \\ |g_2(\mathbf{x}, \mathbf{v}_*) - h_2(\mathbf{x}, \mathbf{v}_*)| & \leq m_{0, \tau}(\mathbf{x}, \mathbf{v}_*) \|m_{0, \tau}^{-1} (g_2 - h_2)\|_{\mathbb{L}^\infty}, \end{aligned}$$

and

$$|g_1(\mathbf{x}, \mathbf{v}) - h_1(\mathbf{x}, \mathbf{v})| \leq m_{0, \tau}(\mathbf{x}, \mathbf{v}) \|m_{0, \tau}^{-1} (g_1 - h_1)\|_{\mathbb{L}^\infty}.$$

Then, due to (33), we find easily

$$\|m_{\alpha, \tau}^{-1} [S_B(g_1, g_2) - S_B(h_1, h_2)]\|_{\mathbb{L}^\infty} \leq \frac{Q_{g_1, g_2}^{h_1, h_2}}{2}.$$

We proceed similarly with (38), by taking advantage of the property  $\mathbf{v}'^2 + \mathbf{v}_*'^2 = \mathbf{v}^2 + \mathbf{v}_*^2$  asserted in (4). We obtain

$$\|m_{\alpha, \tau}^{-1} [P_B(g_1, g_2) - P_B(h_1, h_2)]\|_{\mathbb{L}^\infty} \leq \frac{Q_{g_1, g_2}^{h_1, h_2}}{2}.$$

This completes the proof of a).

b) We introduce inequalities  $|g_1(\mathbf{x}, \mathbf{v})| \leq m_{0, \tau}(\mathbf{x}, \mathbf{v}) \|g_1\|_{\mathbf{B}_\tau}$  and  $|h_2(\mathbf{x}, \mathbf{v})| \leq m_{0, \tau}(\mathbf{x}, \mathbf{v}) \|h_2\|_{\mathbf{B}_\tau}$  in (37). Applying (33) to estimate the integral over  $\mathbf{v}_*$ , and taking the  $\mathbb{L}^1$ -norm, we get

$$\|S_B(g_1, g_2) - S_B(h_1, h_2)\|_{\mathbb{L}^1} \leq \frac{c_{\Lambda, \tau}}{2} (\|g_1\|_{\mathbf{B}_\tau} \|g_2 - h_2\|_{\mathbb{L}^1} + \|h_2\|_{\mathbf{B}_\tau} \|g_1 - h_1\|_{\mathbb{L}^1}).$$

Then, after the standard change of variables  $(\mathbf{v}, \mathbf{v}_*) \rightarrow (\mathbf{v}', \mathbf{v}_*')$  in the integral of (38), similar computations as before give

$$\|P_B(g_1, g_2) - P_B(h_1, h_2)\|_{\mathbb{L}^1} \leq \frac{c_{\Lambda, \tau}}{2},$$

concluding the proof of b).  $\square$

**Corollary 1.** a) For any  $g, h \in \mathbf{B}_\tau$ , one has

$$\|J(g) - J(h)\|_{\mathbf{B}_\tau} \leq c_{\Lambda, \tau} (\|g\|_{\mathbf{B}_\tau} + \|h\|_{\mathbf{B}_\tau}) \|g - h\|_{\mathbf{B}_\tau}, \quad (39)$$

$$\|J^\pi(g) - J^\pi(h)\|_{\mathbf{B}_\tau} \leq c_{\Lambda, \tau} (\|g\|_{\mathbf{B}_\tau} + \|h\|_{\mathbf{B}_\tau}) \|g - h\|_{\mathbf{B}_\tau}. \quad (40)$$

Moreover, for any  $g, h \in \mathbf{M}_\tau$ ,

$$\|J(g) - J(h)\|_{\mathbf{M}_\tau} \leq c_{\Lambda, \tau} (\|g\|_{\mathbf{M}_\tau} + \|h\|_{\mathbf{M}_\tau}) \|g - h\|_{\mathbf{B}_\tau}. \quad (41)$$

b) Let  $0 < \tau < \infty$ . Then, for any  $g, h \in \mathbb{L}^1 \cap \mathbf{B}_\tau$ , one has

$$\|J(g) - J(h)\|_{\mathbb{L}^1} \leq c_{\Lambda, \tau} (\|g\|_{\mathbf{B}_\tau} + \|h\|_{\mathbf{B}_\tau}) \|g - h\|_{\mathbb{L}^1}, \quad (42)$$

$$\|J^\pi(g) - J^\pi(h)\|_{\mathbb{L}^1} \leq c_{\Lambda, \tau} (\|g\|_{\mathbf{B}_\tau} + \|h\|_{\mathbf{B}_\tau}) \|g - h\|_{\mathbb{L}^1}. \quad (43)$$

**Proof:** a) Let  $g_1 = g_2 = h_1 = h_2 = g$  in (35). Then, to obtain (39) and (41), it is sufficient to set  $\alpha = 0$  and  $\alpha = \tau$ , respectively, in (35). Due to (9), inequality (40) follows also from (35) by setting  $g_1 = g$ ,  $g_2 = \pi(g)$ ,  $h_1 = h$ ,  $h_2 = \pi(h)$ ,  $\alpha = 0$ , and applying property (17).

b) To obtain (42), we put  $g_1 = g_2 = h_1 = h_2 = g$  in (36). Finally, due to (9), we get (43) by setting  $g_1 = g$ ,  $g_2 = \pi(g)$ ,  $h_1 = h$ ,  $h_2 = \pi(h)$  in (35) and applying property (17).  $\square$

**Remark 1.** Since  $J(0) = J^\pi(0) = 0$ , it follows that setting  $h = 0$  in each of the inequalities (39)–(43) gives

$$\|J(g)\|_{\mathbf{B}_\tau} \leq c_{\Lambda,\tau} \|g\|_{\mathbf{B}_\tau}^2, \quad \|J^\pi(g)\|_{\mathbf{B}_\tau} \leq c_{\Lambda,\tau} \|g\|_{\mathbf{B}_\tau}^2, \quad (44)$$

$$\|J(g)\|_{\mathbf{M}_\tau} \leq c_{\Lambda,\tau} \|g\|_{\mathbf{M}_\tau} \|g\|_{\mathbf{B}_\tau}, \quad (45)$$

$$\|J(g)\|_{\mathbb{L}^1} \leq c_{\Lambda,\tau} \|g\|_{\mathbf{B}_\tau} \|g\|_{\mathbb{L}^1}, \quad \|J^\pi(g)\|_{\mathbb{L}^1} \leq c_{\Lambda,\tau} \|g\|_{\mathbf{B}_\tau} \|g\|_{\mathbb{L}^1}. \quad (46)$$

With  $\pi$  and  $\Delta x$  defined in Section 2, we have:

**Lemma 2.** For any  $g \in \mathcal{M}_\tau(R, M)$ , one has

$$\|J^\pi(g) - J(g)\|_{\mathbf{B}_\tau} \leq c_{\Lambda,\tau} MR \Delta \mathbf{x}, \quad \Delta \mathbf{x} \leq 1, \quad (47)$$

$$\|J^\pi(g) - J(g)\|_{\mathbb{L}^1} \leq 2\pi^{\frac{9}{2}} b_\Lambda \tau^{-\frac{9-\lambda}{2}} MR \Delta \mathbf{x}, \quad \Delta \mathbf{x} \leq 1. \quad (48)$$

**Proof:** Due to (9), we can set  $g_1 = h_1 = h_2 = g$ ,  $g_2 = \pi g$  and  $\alpha = 0$  in inequality (35). This gives

$$\|J^\pi(g) - J(g)\|_{\mathbf{B}_\tau} \leq c_{\Lambda,\tau} \|g\|_{\mathbf{B}_\tau} \|\pi g - g\|_{\mathbf{B}_\tau}.$$

Similarly, by taking  $g_1 = h_1 = h_2 = g$ ,  $g_2 = \pi g$  in inequality (36), we get

$$\|J^\pi(g) - J(g)\|_{\mathbb{L}^1} \leq c_{\Lambda,\tau} \|g\|_{\mathbf{B}_\tau} \|\pi g - g\|_{\mathbb{L}^1}.$$

To conclude the proof, recall that  $g$  satisfies (20), (24) and (25).  $\square$

Let  $\max(x, y)$  denote the maximum between the real numbers  $x$  and  $y$ .

**Lemma 3.** For any  $g \in \mathcal{M}_\tau(R, M)$ , one has

$$\|U^s g - g\|_{\mathbf{B}_\sigma} \leq 2^{\frac{1}{2}} e^{-\frac{1}{2}} (\tau - \sigma)^{-\frac{1}{2}} \max(R, M) |s|, \quad 0 \leq \sigma < \tau, \quad (49)$$

and

$$\|U^s g - g\|_{\mathbb{L}^1} \leq 4\pi^{\frac{5}{2}} \tau^{-\frac{7}{2}} \max(R, M) |s|. \quad (50)$$

**Proof:** If  $|s\mathbf{v}| \leq 1$ , by applying (19) in (21), we can write

$$|(U^s g)(\mathbf{x}, \mathbf{v}) - g(\mathbf{x}, \mathbf{v})| \leq M |s| |\mathbf{v}| \exp(-\tau \mathbf{v}^2) \exp(-\tau \mathbf{x}^2).$$

Otherwise, we use (19) and (20) in the trivial inequality  $|(U^s g)(\mathbf{x}, \mathbf{v}) - g(\mathbf{x}, \mathbf{v})| \leq |(U^s g)(\mathbf{x}, \mathbf{v})| + |g(\mathbf{x}, \mathbf{v})|$ , and multiplying by  $|s\mathbf{v}|$ , we get

$$\begin{aligned} & |(U^s g)(\mathbf{x}, \mathbf{v}) - g(\mathbf{x}, \mathbf{v})| \\ & \leq R |s| |\mathbf{v}| \exp(-\tau \mathbf{v}^2) [\exp(-\tau (\mathbf{x} - s\mathbf{v})^2) + \exp(-\tau \mathbf{x}^2)], \quad |s\mathbf{v}| > 1. \end{aligned}$$

We can combine the above inequalities into

$$\begin{aligned} & |(U^s g)(\mathbf{x}, \mathbf{v}) - g(\mathbf{x}, \mathbf{v})| \\ & \leq \max(R, M) |s| |\mathbf{v}| \exp(-\tau \mathbf{v}^2) [\exp[-\tau(\mathbf{x} - s\mathbf{v})^2] + \exp(-\tau \mathbf{x}^2)], \quad s \in \mathbb{R}, \end{aligned} \quad (51)$$

for almost all  $(\mathbf{x}, \mathbf{v})$ . To obtain (49), one takes the  $\|\cdot\|_{\mathbf{B}_\sigma}$  - norm of (51) and applies the inequality  $|\mathbf{v}| \exp(\sigma \mathbf{v}^2) \exp(-\tau \mathbf{v}^2) \leq [2e(\tau - \sigma)]^{-\frac{1}{2}}$ . Inequality (50) follows from the  $\mathbb{L}^1$  - norm integration of (51).  $\square$ .

We can now prove the key result of this subsection.

Define

$$I(g, h, t, s) := J^\pi(U^t g) - U^s J(h), \quad \forall g, h \in \mathbf{B}_\tau, \quad t, s \in \mathbb{R}. \quad (52)$$

**Lemma 4.** *Let  $0 < \sigma < \tau < \infty$ . There are some constants  $k_i > 0$ ,  $i = 1, 2$  (depending on  $R, M, \tau, \sigma$  and  $\Lambda$ ), such that:*

a) *For any  $g \in \mathbf{B}_\sigma$ ,  $h, \hat{h} \in \mathcal{M}_\tau(R, M)$  and  $\Delta \mathbf{x} \leq 1$ , one has*

$$\|I(g, h, t, s)\|_{\mathbf{B}_\sigma} \leq c_{\Lambda, \sigma} (\|g\|_{\mathbf{B}_\sigma} + R) \|g - \hat{h}\|_{\mathbf{B}_\sigma} + k_1 (\|\hat{h} - h\|_{\mathbf{B}_\sigma} + \Delta \mathbf{x} + |t| + |s|). \quad (53)$$

b) *For any  $g \in \mathbb{L}^1 \cap \mathbf{B}_\sigma$ ,  $h, \hat{h} \in \mathcal{M}_\tau(R, M)$  and  $\Delta \mathbf{x} \leq 1$ , one has*

$$\|I(g, h, t, s)\|_{\mathbb{L}^1} \leq c_{\Lambda, \sigma} (\|g\|_{\mathbf{B}_\sigma} + R) \|g - \hat{h}\|_{\mathbb{L}^1} + k_2 (\|\hat{h} - h\|_{\mathbb{L}^1} + \Delta \mathbf{x} + |t| + |s|). \quad (54)$$

**Proof:** We write  $I(g, h, t, s) = \sum_{i=1}^5 I_i$ , where  $I_1 = J^\pi(U^t g) - J^\pi(U^t \hat{h})$ ,  $I_2 = J^\pi(U^t \hat{h}) - J^\pi(\hat{h})$ ,  $I_3 = J^\pi(\hat{h}) - J(\hat{h})$ ,  $I_4 = J(\hat{h}) - U^s J(\hat{h})$ , and  $I_5 := U^s J(\hat{h}) - U^s J(h)$ .

We proceed to establishing norm estimates for each  $I_i$ , keeping in mind that, as elements of  $\mathcal{M}_\tau(R, M)$ ,  $h$  and  $\hat{h}$  satisfy (22), and also belong to  $\mathbf{B}_\sigma \cap \mathbb{L}^1$ .

First, we apply inequality (40) to  $I_1$  and  $I_2$ . Then, making use of (15), we get for  $g \in \mathbf{B}_\sigma$ ,

$$\|I_1\|_{\mathbf{B}_\sigma} \leq c_{\Lambda, \sigma} (\|g\|_{\mathbf{B}_\sigma} + R) \|g - \hat{h}\|_{\mathbf{B}_\sigma} \quad (55)$$

and

$$\|I_2\|_{\mathbf{B}_\sigma} \leq 2c_{\Lambda, \sigma} R \|U^t \hat{h} - \hat{h}\|_{\mathbf{B}_\sigma}. \quad (56)$$

Moreover, starting from (43) applied to  $I_1$  and  $I_2$ , and using again (15), we obtain for  $g \in \mathbf{B}_\sigma \cap \mathbb{L}^1$ ,

$$\|I_1\|_{\mathbb{L}^1} \leq c_{\Lambda, \sigma} (\|g\|_{\mathbf{B}_\sigma} + R) \|g - \hat{h}\|_{\mathbb{L}^1} \quad (57)$$

and

$$\|I_2\|_{\mathbb{L}^1} \leq 2c_{\Lambda, \sigma} R \|U^t \hat{h} - \hat{h}\|_{\mathbb{L}^1}. \quad (58)$$

Then, by applying (49) to (56), and (50) to (58), respectively, we obtain

$$\|I_2\|_{\mathbf{B}_\sigma} \leq k_{1,2} |t| \quad (59)$$

and

$$\|I_2\|_{\mathbb{L}^1} \leq k_{2,2} |t|, \quad (60)$$

respectively, where  $k_{1,2} = 2^{\frac{5}{2}} \pi^{\frac{3}{2}} e^{-\frac{1}{2}} b_\Lambda (\tau - \sigma)^{-\frac{1}{2}} \sigma^{-\frac{3-\lambda}{2}} R \max(R, M)$  and  $k_{2,2} = (2\pi)^4 b_\Lambda \tau^{-\frac{7}{2}} \sigma^{-\frac{3-\lambda}{2}} R \max(R, M)$ .

We estimate the norms of  $I_3$ , by means of (47) and (48). We get

$$\|I_3\|_{\mathbf{B}_\sigma} \leq k_{1,3} \Delta \mathbf{x}, \quad \Delta \mathbf{x} \leq 1, \quad (61)$$

with  $k_{1,3} = c_{\Lambda,\sigma}MR$ , and

$$\|I_3\|_{\mathbb{L}^1} \leq k_{2,3}\Delta\mathbf{x}, \quad \Delta\mathbf{x} \leq 1, \quad (62)$$

with  $k_{2,3} = 2\pi^{\frac{9}{2}}b_{\Lambda}\tau^{-\frac{9-\lambda}{2}}MR$ .

To estimate  $I_4$ , observe that since  $\hat{h} \in \mathcal{M}_{\tau}(R, M)$ , then (45) implies  $\|J(\hat{h})\|_{\mathbf{M}_{\tau}} \leq c_{\Lambda,\tau}R^2$ . Moreover, by (41) and the commutation property  $T_{\mathbf{y}}J(\hat{h}) = J(T_{\mathbf{y}}\hat{h})$ , it follows that

$$\|T_{\mathbf{y}}J(\hat{h}) - J(\hat{h})\|_{\mathbf{M}_{\tau}} \leq c_{\Lambda,\tau}(\|(T_{\mathbf{y}}\hat{h})\|_{\mathbf{M}_{\tau}} + \|\hat{h}\|_{\mathbf{M}_{\tau}})\|T_{\mathbf{y}}\hat{h} - \hat{h}\|_{\mathbf{B}_{\tau}}.$$

Suppose that  $|\mathbf{y}| \leq 1$ . Then  $\|T_{\mathbf{y}}\hat{h} - \hat{h}\|_{\mathbf{B}_{\tau}} \leq M$ , because of (21). But  $\|\hat{h}\|_{\mathbf{M}_{\tau}} \leq R$ . Then, clearly,  $\|T_{\mathbf{y}}\hat{h}\|_{\mathbf{M}_{\tau}} \leq (R + M)$ . It follows that

$$\|T_{\mathbf{y}}J(\hat{h}) - J(\hat{h})\|_{\mathbf{M}_{\tau}} \leq c_{\Lambda,\tau}M(2R + M)|\mathbf{y}|, \quad |\mathbf{y}| \leq 1.$$

It appears that  $J(\hat{h}) \in \mathcal{M}_{\tau}(R_*, M_*)$  with  $R_* = c_{\Lambda,\tau}R^2$  and  $M_* = c_{\Lambda,\tau}M(2R + M)$ . Therefore, we can apply Lemma 3 to obtain

$$\|I_4\|_{\mathbf{B}_{\sigma}} \leq k_{1,4}|s|, \quad (63)$$

with  $k_{1,4} = 2^{\frac{3}{2}}e^{-\frac{1}{2}}\pi^{\frac{3}{2}}b_{\Lambda}\tau^{-\frac{3-\lambda}{2}}(\tau - \sigma)^{-\frac{1}{2}}\max(R^2, M(2R + M))$ , and

$$\|I_4\|_{\mathbb{L}^1} \leq k_{2,4}|s|. \quad (64)$$

with  $k_{2,4} = 8\pi^4b_{\Lambda}\tau^{-\frac{10-\lambda}{2}}\max(R^2, M(2R + M))$ .

Finally, by means of (39) and (15),

$$\|I_5\|_{\mathbf{B}_{\sigma}} \leq k_{1,5}\|\hat{h} - h\|_{\mathbf{B}_{\sigma}}, \quad (65)$$

while, by (42) and (15),

$$\|I_5\|_{\mathbb{L}^1} \leq k_{2,5}\|\hat{h} - h\|_{\mathbb{L}^1}, \quad (66)$$

with  $k_{1,5} = k_{2,5} = 2c_{\Lambda,\sigma}R$ .

Inequalities (53) and (54) follow directly from the above estimates, with  $k_i = \max_{2 \leq j \leq 5}(k_{i,j})$ ,  $i = 1, 2$ .  $\square$

We end this section with a useful lemma on the mild solutions of (1).

**Lemma 5.** *Let  $f(t)$  be a mild solution of Eq. (1), and  $f(t) \in \mathcal{M}_{\tau}(R, M)$  for all  $0 \leq t \leq T$ .*

a) *Let  $0 < \sigma < \tau$ . Then for each  $t, s \in [0, T]$ ,*

$$\|f(t) - f(s)\|_{\mathbf{B}_{\sigma}} \leq d_1|t - s|, \quad (67)$$

where  $d_1 = 2^{\frac{1}{2}}e^{-\frac{1}{2}}(\tau - \sigma)^{-\frac{1}{2}}\max(R, M) + c_{\Lambda,\sigma}R^2$ .

b) *For each  $t, s \in [0, T]$ ,*

$$\|f(t) - f(s)\|_{\mathbb{L}^1} \leq d_2|t - s|, \quad (68)$$

where  $d_2 = 4\pi^{\frac{5}{2}}\tau^{-\frac{7}{2}}\max(R, M) + c_{\Lambda,\sigma}\left(\frac{\pi}{\sigma}\right)^3R^2$ .

**Proof:** We can suppose  $t \geq s$ . From Eq. (26),

$$f(t) - f(s) = U^{t-s}f(s) - f(s) + \int_s^t U^{t-u}J(f(u))du. \quad (69)$$

a) Taking the  $\mathbf{B}_{\sigma}$  norm in (69) and using (15), we have

$$\|f(t) - f(s)\|_{\mathbf{B}_{\sigma}} \leq \|U^{t-s}f(s) - f(s)\|_{\mathbf{B}_{\sigma}} + \int_s^t \|J(f(u))\|_{\mathbf{B}_{\sigma}}du. \quad (70)$$

Further, we apply (49) to estimate the first term in (70). We obtain

$$\|U^{t-s}f_0 - f_0\|_{\mathbf{B}_\sigma} \leq 2^{\frac{1}{2}}e^{-\frac{1}{2}}(\tau - \sigma)^{-\frac{1}{2}}\max(R, M)(t - s). \quad (71)$$

Furthermore, we estimate the integral term of (70) by means of (44) and applying property (22) to  $f$ . We obtain

$$\int_s^t \|J(f(u))\|_{\mathbf{B}_\sigma} du \leq c_{\Lambda, \sigma} R^2(t - s). \quad (72)$$

Now (67) results by introducing (71) and (72) in (70).

b) Taking the  $\mathbb{L}^1$  norm in (69) and using (15), we get

$$\|f(t) - f(s)\|_{\mathbb{L}^1} \leq \|U^{t-s}f(s) - f(s)\|_{\mathbb{L}^1} + \int_s^t \|J(f(u))\|_{\mathbb{L}^1} du. \quad (73)$$

Using (50) to estimate the first term in the r.h.s. of (73), we get

$$\|U^{t-s}f(s) - f(s)\|_{\mathbb{L}^1} \leq 4\pi^{\frac{5}{2}}\tau^{-\frac{7}{2}}\max(R, M)(t - s). \quad (74)$$

To estimate the second term in the r.h.s. of (73), we apply (46) and the fact that  $f$  satisfies both (22) and (23). We obtain

$$\int_s^t \|J(f(u))\|_{\mathbb{L}^1} du \leq c_{\Lambda, \sigma} \left(\frac{\pi}{\sigma}\right)^3 R^2(t - s). \quad (75)$$

Finally, (68) is a consequence of (74) and (75) introduced in (73).  $\square$

**4.2. Uniform  $\mathbf{B}$  - norm boundedness of  $(\tilde{f}^j)_j$ . Proof of Theorem 1.** In the following, we suppose that the assumptions of Theorem 1 are satisfied. Thus, under the conditions of Theorem 1, the mild solution  $f(t)$  of the Cauchy problem (1) satisfies

$$f(t) \in \mathcal{M}_\tau(R, M) \subset \mathbf{B}_\tau \subset \mathbf{B}_\sigma \quad (76)$$

and

$$\|f(t)\|_{\mathbf{B}_\sigma} \leq \|f(t)\|_{\mathbf{B}_\tau} \leq \|f(t)\|_{\mathbf{M}_\tau} \leq R, \quad (77)$$

for all  $0 \leq t \leq T$ ,  $0 < \sigma < \tau$ .

Put simply, the central (convergence) part of the argument behind Theorem 1 consists in obtaining an appropriate uniform  $\mathbb{L}^1$  - norm estimate for the difference  $\tilde{f}^j - f(t_j)$ , where  $\tilde{f}^j$  is given by the recurrence (7), and  $f(t_j)$  is the solution of Eq. (26) at moment  $t_j := j\Delta t$ ,  $j = 0, 1, \dots, \lceil T/\Delta t \rceil$ . To this end, one first needs to establish the boundedness of the sequence  $(\tilde{f}^j)_j$  in a suitable  $\mathbf{B}$  - norm. Technically, the proof of Theorem 1 applies Lemma 4. Thus, since  $f_0$  satisfies (76), a straightforward induction (based on the application of (15) and (44) to the recurrence (7)) implies

$$\tilde{f}^j \in \mathbf{B}_\sigma. \quad (78)$$

Consequently, (26) and (7) can be combined in a well-defined (at least in  $\mathbf{B}_\sigma$ ) expression

$$\tilde{f}^j - f(t_j) = U^{\Delta t}(\tilde{f}^{j-1} - f(t_{j-1})) + \int_{t_{j-1}}^{t_j} [J^\pi(U^{\Delta t}\tilde{f}^{j-1}) - U^{t_j-u}J(f(u))]du \quad (79)$$

( $j = 1, 2, \dots, \lfloor T/\Delta t \rfloor$ ). In essence, the central estimates of the proof of Theorem 1 are obtained by applying Lemma 4 to (79), based on the immediate observation that the integrand of (79) satisfies

$$J^\pi(U^{\Delta t} \tilde{f}^{j-1}) - U^{t_j-u} J(f(u)) = I(\tilde{f}^{j-1}, f(u), \Delta t, t_j - u), \quad (80)$$

where  $I$  is defined by (52).

In detail, we first prove the following proposition which provides the aforementioned  $\mathbf{B}$  - norm boundedness of the sequence  $(\tilde{f}^j)_j$  (and also yields the convergence of  $(\tilde{f}^j)_j$  with respect to the  $\mathbf{B}$  - norm).

For some  $0 < \sigma < \tau$ , define

$$\tilde{F}^j := \max\{\|\tilde{f}^0\|_{\mathbf{B}_\sigma}, \|\tilde{f}^1\|_{\mathbf{B}_\sigma}, \dots, \|\tilde{f}^{j-1}\|_{\mathbf{B}_\sigma}\}, \quad j = 1, 2, \dots, \lfloor T/\Delta t \rfloor. \quad (81)$$

Due to (78), obviously,  $\tilde{F}^j < \infty$ ,  $j = 1, 2, \dots, \lfloor T/\Delta t \rfloor$ .

**Proposition 1.** *a) There exists a strictly increasing continuous function  $C(\cdot) : [0, \infty) \rightarrow (0, \infty)$  (parameterized by  $R, M, T, \tau, \sigma$  and  $\Lambda$ ) such that for any recurrence of the form (7), with time-step  $0 < \Delta t < T$  and cell-partition diameter  $0 < \Delta \mathbf{x} < 1$ ,*

$$\|\tilde{f}^j - f(t_j)\|_{\mathbf{B}_\sigma} \leq C(\tilde{F}^j)(\Delta t + \Delta \mathbf{x}), \quad j = 1, 2, \dots, \lfloor T/\Delta t \rfloor. \quad (82)$$

*b) There exist some numbers  $\rho > 0$ ,  $0 < X_0 < 1$  and  $0 < T_0 < \min(1, T)$  (depending on  $R, M, T, \tau, \sigma$  and  $\Lambda$ ) such that, for any recurrence (7) with  $0 < \Delta \mathbf{x} \leq X_0$  and  $0 < \Delta t \leq T_0$ ,*

$$\|\tilde{f}^j - f(t_j)\|_{\mathbf{B}_\sigma} \leq C(R + \rho)(\Delta t + \Delta \mathbf{x}) \quad (83)$$

and

$$\|\tilde{f}^j\|_{\mathbf{B}_\sigma} \leq R + \rho, \quad (84)$$

$j = 1, 2, \dots, \lfloor T/\Delta t \rfloor$ .

**Proof:** a) As  $\tilde{f}^j \in \mathbf{B}_\sigma$ , one can apply Lemma 4 a), with  $\hat{h} = f(t_{j-1})$ , to (80). The resulting inequality contains the expression  $\|f(t_{j-1}) - f(u)\|_{\mathbf{B}_\sigma}$  which is then estimated by Lemma 5 a). One finds

$$\begin{aligned} \|J^\pi(U^{\Delta t} \tilde{f}^{j-1}) - U^{t_j-u} J(f(u))\|_{\mathbf{B}_\sigma} &\leq c_{\Lambda, \sigma}(\|\tilde{f}^{j-1}\|_{\mathbf{B}_\sigma} + R) \|\tilde{f}^{j-1} - f(t_{j-1})\|_{\mathbf{B}_\sigma} \\ &+ k_1[\Delta \mathbf{x} + \Delta t + (1 + d_1)(t_j - u)], \end{aligned} \quad (85)$$

where the constants  $k_1$  and  $d_1$  are given by Lemmas 4 a) and 5 a), respectively. By applying (85) to estimate the  $\mathbf{B}_\sigma$  - norm of (79), we obtain that there is some number  $0 < k < \infty$  (depending on  $R, M, \tau, \sigma$  and  $\Lambda$ ) such that

$$\begin{aligned} \|\tilde{f}^j - f(t_j)\|_{\mathbf{B}_\sigma} &\leq [1 + \varphi(\|\tilde{f}^{j-1}\|_{\mathbf{B}_\sigma})\Delta t] \|\tilde{f}^{j-1} - f(t_{j-1})\|_{\mathbf{B}_\sigma} \\ &+ k\Delta t(\Delta t + \Delta \mathbf{x}), \quad j = 1, 2, \dots, \lfloor T/\Delta t \rfloor, \end{aligned} \quad (86)$$

where  $\varphi(x) := c_{\Lambda, \sigma}(x + R)$ .

Fix some  $j^* = 1, 2, \dots, \lfloor T/\Delta t \rfloor$ . Due to (81) and the monotonicity of  $\varphi$ , in (86), we can apply the inequality  $\varphi(\tilde{f}^j) \leq \varphi(\tilde{F}^{j^*})$ ,  $j = 0, 1, \dots, j^* - 1$ . We are thus led to the following simple Gronwall - type discrete scheme

$$\begin{aligned} \|\tilde{f}^j - f(t_j)\|_{\mathbf{B}_\sigma} &\leq [1 + \varphi(\tilde{F}^{j^*})\Delta t] \|\tilde{f}^{j-1} - f(t_{j-1})\|_{\mathbf{B}_\sigma} \\ &+ k\Delta t(\Delta t + \Delta \mathbf{x}), \quad j = 1, 2, \dots, j^*. \end{aligned} \quad (87)$$

As  $\tilde{f}^0 = f(0)$ , by iterating (87), we get

$$\|\tilde{f}^{j^*} - f(t_{j^*})\|_{\mathbf{B}_\sigma} \leq [1 + \varphi(\tilde{F}^{j^*})\Delta t]^{j^*} \frac{k(\Delta t + \Delta \mathbf{x})}{\varphi(\tilde{F}^{j^*})}.$$

However,  $j^* \leq T/\Delta t$ . Consequently,

$$\begin{aligned} \|\tilde{f}^{j^*} - f(t_{j^*})\|_{\mathbf{B}_\sigma} &\leq [1 + \varphi(\tilde{F}^{j^*})\Delta t]^{\frac{T}{\Delta t}} \frac{k(\Delta t + \Delta \mathbf{x})}{\varphi(\tilde{F}^{j^*})} \\ &\leq \frac{k \exp[(\varphi(\tilde{F}^{j^*})T)]}{\varphi(\tilde{F}^{j^*})} (\Delta t + \Delta \mathbf{x}). \end{aligned}$$

Since  $\varphi(\tilde{F}^{j^*}) \geq c_{\Lambda, \sigma} R$ , it follows that

$$\|\tilde{f}^{j^*} - f(t_{j^*})\|_{\mathbf{B}_\sigma} \leq C(\tilde{F}^{j^*})(\Delta t + \Delta \mathbf{x}),$$

with

$$C(x) := \frac{k}{c_{\Lambda, \sigma} R} \exp[c_{\Lambda, \sigma} T(x + R)] > 0, \quad (88)$$

which is strictly increasing in  $x$  on  $[0, \infty)$ . This concludes the proof of a), because  $j^* \leq \lceil T/\Delta t \rceil$  is arbitrary.

b) From (88) it follows that there is  $\rho > 0$  such that  $0 < \rho/C(R + \rho) < \min(1, T)$ . Let  $0 < X_0 < \rho/C(R + \rho)$  and  $T_0 = \rho/C(R + \rho) - X_0$ . Therefore, for any  $\Delta \mathbf{x} \leq X_0$  and  $\Delta t \leq T_0$ , we have

$$0 < C(R + \rho)(\Delta t + \Delta \mathbf{x}) \leq C(R + \rho)(T_0 + X_0) \leq \rho. \quad (89)$$

As  $\|\tilde{f}^0\|_{\mathbf{B}_\sigma} = \|f_0\|_{\mathbf{B}_\sigma} \leq R + \rho$ , we get  $\|\tilde{f}^1 - f(t_1)\|_{\mathbf{B}_\sigma} \leq C(R + \rho)(\Delta t + \Delta \mathbf{x}) \leq \rho$ , by virtue of (82) and (89). Since  $f(t_1)$  satisfies (77), it follows that  $\|\tilde{f}^1\|_{\mathbf{B}_\sigma} \leq R + \rho$ . Then a straightforward induction concludes the proof of b).  $\square$

Based on Proposition 1, we can now prove the main result of the paper.

**Proof of Theorem 1:** With the notations of the theorem, let  $0 < \sigma < \tau$ .

a) Proof of (27): First we show that  $f_{\Delta t, \Delta \mathbf{x}}(t) \in \mathbb{L}^1 \cap \mathbf{B}_\sigma$ .

By virtue of (78), we need only prove that

$$\tilde{f}^j \in \mathbb{L}^1, \quad j = 0, 1, \dots, \lceil T/\Delta t \rceil. \quad (90)$$

Obviously,  $\tilde{f}^0 = f_0 \in \mathcal{M}_\tau(R, M) \subset \mathbb{L}^1$ . To check that  $\tilde{f}^1 \in \mathbb{L}^1$ , first observe that from (79), particularized to  $j = 1$ , one finds

$$\|\tilde{f}^1\|_{\mathbb{L}^1} \leq \|f(t_1)\|_{\mathbb{L}^1} + \int_0^{\Delta t} (\|J^\pi(U^{\Delta t} f_0)\|_{\mathbb{L}^1} + \|U^{\Delta t - u} J(f(u))\|_{\mathbb{L}^1}) du. \quad (91)$$

Since  $f(t) \in \mathcal{M}_\tau(R, M) \subset \mathbb{L}^1$ , we need only check that the integral term of the above inequality is finite. To this end, we apply (46) and (15), to estimate the terms of the sum under the integral in (91). We get  $\|J^\pi(U^{\Delta t} f_0)\|_{\mathbb{L}^1} \leq c_{\Lambda, \sigma} \|f_0\|_{\mathbb{B}_\sigma} \|f_0\|_{\mathbb{L}^1}$  and  $\|U^{\Delta t - u} J(f(u))\|_{\mathbb{L}^1} \leq c_{\Lambda, \sigma} \|f(u)\|_{\mathbf{B}_\sigma} \|f(u)\|_{\mathbb{L}^1}$ . Now it remains to observe that  $\|f(u)\|_{\mathbb{L}^1}$  satisfies (23), and that  $\|f(u)\|_{\mathbf{B}_\sigma} \leq R$ , by virtue of (77).

As  $\tilde{f}^j$  satisfies (84), the proof of (90) is completed by induction, following a similar argument as before, based on the application of (46) and (15).

To conclude the proof of (27), it remains to show that  $\tilde{f}^j \geq 0$ . We proceed by induction, applying a trick as in [6].

1)  $f^0 \geq 0$  by hypothesis.

2) By (7),

$$\tilde{f}^j = U^{\Delta t} \tilde{f}^{j-1} - \Delta t S^\pi(U^{\Delta t} \tilde{f}^{j-1}) + \Delta t P^\pi(U^{\Delta t} \tilde{f}^{j-1}), \quad j = 1, \dots, \lfloor T/\Delta t \rfloor.$$

Suppose that  $\tilde{f}^{j-1} \geq 0$ . As  $U^{\Delta t}$  and  $P^\pi$  are positivity preserving operators, in order to show that  $\tilde{f}^j \geq 0$ , it is sufficient to prove that

$$U^{\Delta t} \tilde{f}^{j-1} - \Delta t S^\pi(U^{\Delta t} \tilde{f}^{j-1}) \geq 0. \quad (92)$$

Observe that

$$U^{\Delta t} \tilde{f}^{j-1} - \Delta t S^\pi(U^{\Delta t} \tilde{f}^{j-1}) = U^{\Delta t} \tilde{f}^{j-1} [1 - \Delta t E(U^{\Delta t} \tilde{f}^{j-1})],$$

where

$$E(g)(\mathbf{x}, \mathbf{v}) := \sum_{l \in \mathbb{N}} \chi_l(\mathbf{x}) \frac{1}{|\pi_l|} \int_{\pi_l} d\mathbf{y} \int_{\mathbb{R}^3 \times \mathbb{S}^2} d\mathbf{v}_* d\boldsymbol{\omega} b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\omega}) g(\mathbf{y}, \mathbf{v}_*).$$

But (84) gives

$$(U^{\Delta t} \tilde{f}^{j-1})(\mathbf{y}, \mathbf{v}_*) < (R + \rho) \exp(-\sigma |\mathbf{v}_*|^2), \quad j = 1, \dots, \lfloor T/\Delta t \rfloor,$$

for  $0 < \Delta \mathbf{x} \leq X_0$  and  $0 < \Delta t \leq T_0$  as in Proposition 1. Then, by virtue of (33),

$$E(U^{\Delta t} \tilde{f}^{j-1}) \leq \frac{1}{2} c_{\Lambda, \sigma} (R + \rho) =: \mathcal{D}_0, \quad j = 1, \dots, \lfloor T/\Delta t \rfloor.$$

Therefore, it is sufficient to set  $T_* = \min(T_0, \mathcal{D}_0^{-1})$ , in order that the inequality (92) be satisfied.

b) Proof of (28): We show that there is some number  $K > 0$  (depending on  $R, M, T, \tau, \sigma$  and  $\Lambda$ ) such that for all  $j = 0, 1, \dots, \lfloor T/\Delta t \rfloor$ ,

$$\|\tilde{f}^j - f(t_j)\|_{\mathbb{L}^1} \leq K(\Delta t + \Delta \mathbf{x}), \quad 0 < \Delta t \leq T_*, \quad 0 < \Delta \mathbf{x} \leq X_*, \quad (93)$$

with  $X_* = X_0$ , where  $X_0$  is as in Proposition 1.

We start as in the proof Proposition 1 a). Since  $\tilde{f}^j \in \mathbb{L}^1 \cap \mathbf{B}_\sigma$ , we apply Lemma 4 b), with  $\hat{h} = f(t_{j-1})$ , to (80). The resulting inequality contains the expression  $\|f(t_{j-1}) - f(u)\|_{\mathbb{L}^1}$  which is then estimated by Lemma 5 b). We obtain

$$\begin{aligned} & \|J^\pi(U^{\Delta t} \tilde{f}^{j-1}) - U^{t_j - u} J(f(u))\|_{\mathbb{L}^1} \leq c_{\Lambda, \sigma} (\|\tilde{f}^{j-1}\|_{\mathbf{B}_\sigma} + R) \\ & \times \|\tilde{f}^{j-1} - f(t_{j-1})\|_{\mathbb{L}^1} + k_2 [\Delta \mathbf{x} + \Delta t + (1 + d_2)(t_j - u)], \end{aligned} \quad (94)$$

where the constants  $k_2$  and  $d_2$  are given by Lemmas 4 b) and 5 b), respectively. We apply (94) to estimate the  $\mathbb{L}^1$  - norm of (79). Also, we take advantage of (15) and of the key property (84). After a straightforward computation, it follows that there exist two constants  $\tilde{K}_i > 0$ ,  $i = 1, 2$  (which depend on  $R, M, T, \tau, \sigma$  and  $\Lambda$ ) such that

$$\begin{aligned} & \|\tilde{f}^j - f(t_j)\|_{\mathbb{L}^1} \leq (1 + \tilde{K}_1 \Delta t) \|\tilde{f}^{j-1} - f(t_{j-1})\|_{\mathbb{L}^1} \\ & + \tilde{K}_2 \Delta t (\Delta t + \Delta \mathbf{x}), \quad j = 1, 2, \dots, \lfloor T/\Delta t \rfloor. \end{aligned} \quad (95)$$

The simple scheme (95) can be iterated directly with respect to  $\|\tilde{f}^j - f(t_j)\|_{\mathbb{L}^1}$ . As  $\tilde{f}^0 = f(0)$ , we get

$$\|\tilde{f}^j - f(t_j)\|_{\mathbb{L}^1} \leq \frac{\tilde{K}_2}{\tilde{K}_1} (1 + \tilde{K}_1 \Delta t)^j (\Delta t + \Delta \mathbf{x}).$$



However,  $j \leq \lceil T/\Delta t \rceil$ . Therefore,

$$\|\tilde{f}^j - f(t_j)\|_{\mathbb{L}^1} \leq \frac{\tilde{K}_2}{\tilde{K}_1} (1 + \tilde{K}_1 \Delta t)^{\frac{T}{\Delta t}} (\Delta t + \Delta \mathbf{x}) \leq \frac{\tilde{K}_2}{\tilde{K}_1} \exp(\tilde{K}_1 T) (\Delta t + \Delta \mathbf{x}),$$

for all  $j = 1, 2, \dots, \lceil T/\Delta t \rceil$ . Finally, set

$$K = \frac{\tilde{K}_2}{\tilde{K}_1} \exp(\tilde{K}_1 T),$$

we obtain (93).

Now (28) follows directly from (93) and (68), with  $K_1 = d_2 + K$  and  $K_2 = K$ , where  $d_2$  is the constant of (68).

c) Proof of (14): The conservation property (14) follows from (6) and (7), by observing that, by virtue of (2) and (10), one obtains

$$\begin{aligned} & \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_i(\mathbf{v}) J_B(U^{\Delta t} \tilde{f}^{j-1}, \pi U^{\Delta t} \tilde{f}^{j-1})(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} \\ &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_i(\mathbf{v}) \pi J_B(U^{\Delta t} \tilde{f}^{j-1}, \pi U^{\Delta t} \tilde{f}^{j-1})(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} \\ &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_i(\mathbf{v}) J_B(\pi U^{\Delta t} \tilde{f}^{j-1}, \pi U^{\Delta t} \tilde{f}^{j-1})(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v}, \quad i = 0, 1, 2, 3, 4, \end{aligned}$$

with  $\varphi_i(\mathbf{v})$  as in (6). Then it is sufficient to invoke (6).  $\square$

**4.3. Proof of Theorem 2.** Since  $\|f(t)\|_{\mathbf{M}_\tau} \leq \|f(t)\|_{\mathbf{M}_{\tau_*}} \leq R$  for all  $0 \leq t \leq T$ ,  $0 < \tau \leq \tau_*$ , it is sufficient to prove that  $f(t)$  satisfies an inequality of the form (21).

We start with the remark that since  $T_{\mathbf{y}}$  commutes with  $U^t$  and  $J$ , then, by virtue of (26), we have, for any  $0 \leq t \leq T$ ,

$$T_{\mathbf{y}} f(t) - f(t) = U^t(T_{\mathbf{y}} f_0 - f_0) + \int_0^t U^{t-s} [J(T_{\mathbf{y}} f(s)) - J(f(s))] ds. \quad (96)$$

Let  $0 < \tau_1 < \tau_*$ . Due to (16), there is some  $0 < \tau < \tau_1$ , such that

$$\|T_{\mathbf{y}} f(t) - f(t)\|_{\mathbf{M}_\tau} \leq \|T_{\mathbf{y}} f_0 - f_0\|_{\mathbf{M}_{\tau_1}} + \int_0^t \|J(T_{\mathbf{y}} f(s)) - J(f(s))\|_{\mathbf{M}_{\tau_1}} ds. \quad (97)$$

Observing that the first term of the sum in (97) satisfies

$$\|(T_{\mathbf{y}} f_0 - f_0)\|_{\mathbf{M}_{\tau_1}} \leq \|(T_{\mathbf{y}} f_0 - f_0)\|_{\mathbf{M}_{\tau_*}} \leq M_0 |\mathbf{y}|, \quad |\mathbf{y}| \leq 1,$$

and introducing (41) in the integral term of (97), we obtain

$$\begin{aligned} & \|T_{\mathbf{y}} f(t) - f(t)\|_{\mathbf{M}_\tau} \leq M_0 |\mathbf{y}| \\ & + c_{\Lambda, \tau_1} \int_0^t (\|T_{\mathbf{y}} f(s)\|_{\mathbf{M}_{\tau_1}} + \|f(s)\|_{\mathbf{M}_{\tau_1}}) \|T_{\mathbf{y}} f(s) - f(s)\|_{\mathbf{B}_{\tau_1}} ds. \end{aligned} \quad (98)$$

Further, we estimate the factors of the product under the integral sign in (98). To this end, by observing that a straightforward computation gives  $\|T_{\mathbf{y}} f(s)\|_{\mathbf{M}_{\tau_1}} \leq$

$R \exp\left(\frac{\tau_*^2}{\tau_* - \tau_1} x\right)$ , for all  $0 \leq s \leq T$ , and  $|\mathbf{y}| \leq 1$ , and using  $\|f(s)\|_{\mathbf{M}_{\tau_1}} \leq \|f(s)\|_{\mathbf{M}_{\tau_*}} \leq R$ , we get

$$\|T_{\mathbf{y}}f(s)\|_{\mathbf{M}_{\tau_1}} + \|f(s)\|_{\mathbf{M}_{\tau_1}} \leq R \left[1 + \exp\left(\frac{\tau_*^2}{\tau_* - \tau_1}\right)\right], \quad |\mathbf{y}| \leq 1, \quad 0 \leq s \leq T. \quad (99)$$

To estimate the second factor of the product under the integral sign of (98), first observe that

$$\|T_{\mathbf{y}}f_0 - f_0\|_{\mathbf{B}_{\tau_1}} \leq M_0|\mathbf{y}|, \quad |\mathbf{y}| \leq 1, \quad (100)$$

because of the assumption  $f_0 \in \mathbf{M}_{\tau_*}(R, M_0)$ . Then, due to property (100), a standard argument applied to (96) gives

$$\|T_{\mathbf{y}}f(t) - f(t)\|_{\mathbf{B}_{\tau_1}} \leq M_0 \exp(2c_{\Lambda, \tau_1} R t) |\mathbf{y}|, \quad |\mathbf{y}| \leq 1, \quad 0 \leq t \leq T. \quad (101)$$

Indeed, by taking the  $\mathbf{B}_{\tau_1}$ -norm of (96), and applying (15), (100) and (39), we obtain

$$\begin{aligned} & \|T_{\mathbf{y}}f(t) - f(t)\|_{\mathbf{B}_{\tau_1}} \leq M_0|\mathbf{y}| \\ & + c_{\Lambda, \tau_1} \int_0^t (\|T_{\mathbf{y}}f(s)\|_{\mathbf{B}_{\tau_1}} + \|f(s)\|_{\mathbf{B}_{\tau_1}}) \|T_{\mathbf{y}}f(s) - f(s)\|_{\mathbf{B}_{\tau_1}} ds, \quad |\mathbf{y}| \leq 1. \end{aligned}$$

But (18) and (29) imply  $\|T_{\mathbf{y}}f(t)\|_{\mathbf{B}_{\tau_1}} = \|f(t)\|_{\mathbf{B}_{\tau_1}} \leq \|f(t)\|_{\mathbf{M}_{\tau_*}} \leq R$ , hence

$$\|T_{\mathbf{y}}f(t) - f(t)\|_{\mathbf{B}_{\tau_1}} \leq M_0|\mathbf{y}| + 2c_{\Lambda, \tau_1} R \int_0^t \|T_{\mathbf{y}}f(s) - f(s)\|_{\mathbf{B}_{\tau_1}} ds,$$

so that the application of Gronwall's inequality yields (101).

Thus, by using (99) and (101) in (98), we finally obtain

$$\|T_{\mathbf{y}}f(t) - f(t)\|_{\mathbf{M}_{\tau}} \leq M|\mathbf{y}|, \quad |\mathbf{y}| \leq 1, \quad 0 \leq t \leq T,$$

with

$$M = M_0 \left\{ 1 + \frac{1}{2} (\exp(2c_{\Lambda, \tau_1} R T) - 1) \left[ 1 + \exp\left(\frac{\tau_*^2}{\tau_* - \tau_1}\right) \right] \right\}, \quad (102)$$

hence (31) is satisfied with  $M$  of (102). This concludes the proof.  $\square$

## 5. EXAMPLE AND CONCLUSIONS

We present a simple application of Theorem 2 to the solutions of the Cauchy problem for the Boltzmann equation near vacuum. (We skip over mentioning parametric dependencies as not being particularly relevant for our purposes.)

In what follows, it is sufficient to consider the existence and uniqueness of local in time, positive solutions to Eq. (26) for a small initial datum bounded by a space-velocity Maxwellian. The following result can be easily obtained by applying the Kaniel-Shinbrot monotone iteration scheme [8, 15], or by elementary fixed point methods [11].

**Proposition 2.** *Let  $0 < \tau_0 < \infty$  and  $0 \leq f_0 \in \mathbf{M}_{\tau_0}$ . For each  $T > 0$ , there are  $0 < r < R < \infty$  such that if  $\|f_0\|_{\mathbf{M}_{\tau_0}} \leq r$ , then Eq. (26) has a unique solution satisfying*

$$0 \leq f(t, \mathbf{x}, \mathbf{v}) \leq R \exp[-\tau_0(\mathbf{x} - t\mathbf{v})^2 - \tau_0 \mathbf{v}^2], \quad 0 \leq t \leq T, \quad (103)$$

for almost all  $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ .

Our application follows by combining the above proposition with Theorem 2.

**Proposition 3.** *Let  $0 < \tau_0, M_0 < \infty$ . For each  $T > 0$ , there are  $0 < r, R, M < \infty$  and  $0 < \tau < \tau_0$  such that if  $0 \leq f_0 \in \mathcal{M}_{\tau_0}(r, M_0)$ , then the Cauchy problem (1) has a unique mild solution  $0 \leq f(t) \in \mathcal{M}_\tau(R, M)$  for all  $0 \leq t \leq T$ .*

**Proof:** Consider the solution  $f$  of Eq. (26) provided by the above proposition. By virtue of (16), there is  $0 < \Theta = \Theta(T, \tau_0) < \tau_0$  such that for any  $0 \leq \tau_* < \Theta$ , one can write  $\|f(t)\|_{\mathbf{M}_{\tau_*}} = \|U^t U^{-t} f(t)\|_{\mathbf{M}_{\tau_*}} \leq \|U^{-t} f(t)\|_{\mathbf{M}_{\tau_0}}$ ,  $0 \leq t \leq T$ . Then, due to (103),

$$\|f(t)\|_{\mathbf{M}_{\tau_*}} \leq R, \quad 0 \leq t \leq T. \quad (104)$$

Obviously,  $f(t) \in \mathbb{L}^1$ , for all  $t \geq 0$ . Moreover,  $f_0 \in \mathcal{M}_{\tau_*}(R, M_0)$ , because  $f_0 \in \mathcal{M}_{\tau_0}(r, M_0)$ ,  $0 < \tau_* < \tau_0$ , and  $r < R$ . Consequently, Theorem 2 applies, concluding the proof.  $\square$

We end this sections with a few comments about our results.

Theorem 1 implies immediately the convergence in discrepancy of (12). Thus, the validation of Nanbu's simulation scheme for the Boltzmann equation in the whole space and Maxwellian molecular interactions can be supplemented with a similar result as in [1].

To better clarify why the approach of [1] is not directly applicable to our setting (for the Boltzmann equation in the whole space), recall that expression (83) establishes the convergence of the solutions of the recurrence (7) in a suitable  $\mathbf{B}$  - norm, i.e. in a (Maxwellian-weighted)  $\mathbb{L}^\infty$  - space. Such a property was sufficient to ensure the convergence in  $\mathbb{L}^1$ , in the setting of [1] for the Boltzmann gas in a finite domain  $\Omega$ , because of the continuous embedding of the space  $\{h : m_{0,\tau}^{-1} h \in \mathbb{L}^\infty(\Omega \times \mathbb{R}^3; d\mathbf{x}d\mathbf{v})\}$  into  $\mathbb{L}^1(\Omega \times \mathbb{R}^3; d\mathbf{x}d\mathbf{v})$ , which holds when  $\Omega$  is bounded. However, this is not the case if  $\Omega = \mathbb{R}^3$ , when solely the weak convergence property (83) is not sufficient to imply the  $\mathbb{L}^1$  - convergence.

Theorem 2 extends somehow the simpler property, mentioned in [1], that if the initial condition of Eq. (26) satisfies  $f_0 \in \mathcal{B}_\tau(R, M)$ , then the solution of the equation also satisfies  $f(t) \in \mathcal{B}_\tau(R, M)$  for all  $0 < t \leq T$ . Nevertheless, the latter property remains valid in the context of the Boltzmann equation in the entire space, being actually established within the proof of Theorem 2, by deriving inequality (101) as a consequence of (100).

Following a line of reasoning as in the present paper, Theorem 1 can be generalized to a wider class of solutions of the Boltzmann equation, with slower decay at infinity, like those considered in some investigations on the Cauchy problem for the Boltzmann equation with near-vacuum conditions [9, 10].

The results of this paper can be also extended to more complicated Boltzmann like models as those describing several spaces of chemically interacting fluids [11, 16, 17]. A potential application would be the validation of the space-dependent Nanbu scheme for the reacting gas, by extending results obtained in the space-homogeneous case in [18], [19].

Due to the explicit form of the constants involved in the technical inequalities of Section 4, the proofs of Proposition 1 and Theorem 2 may be also detailed to provide explicit upper bounds for the constants  $K_1$  and  $K_2$  of (28), as well as for the other constants appearing in Theorem 2. Such upper bounds may be useful in estimating the errors introduced by the approximation (7), as well as in the

parametric optimization of the approximation. However, a detailed computation of the above bounds is beyond the scope of this paper.

#### ACKNOWLEDGMENT

This work was partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-RU-TE-2012-3-0196

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